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Published in:
Journal of Statistical Mechanics: Theory and Experiment

DOI:
10.1088/1742-5468/2007/08/P08026

Publication date:
2007

Document Version
Publisher's PDF, also known as Version of record

Link to publication

Citation for pulished version (HARVARD):
Lambiotte, R & Ausloos, M 2007, 'Coexistence of opposite opinions in a network with communities', *Journal of Statistical Mechanics: Theory and Experiment*, no. 8. https://doi.org/10.1088/1742-5468/2007/08/P08026

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Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

Coexistence of opposite opinions in a network with communities

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Received 29 March 2007
Accepted 20 July 2007
Published 21 August 2007

Online at stacks.iop.org/JSTAT/2007/P08026
doi:10.1088/1742-5468/2007/08/P08026

Abstract. The majority rule is applied to a topology that consists of two coupled random networks, thereby mimicking the modular structure observed in social networks. We calculate analytically the asymptotic behaviour of the model and derive a phase diagram that depends on the frequency of random opinion flips and on the inter-connectivity between the two communities. It is shown that three regimes may take place: a disordered regime, where no collective phenomena takes place; a symmetric regime, where the nodes in both communities reach the same average opinion; and an asymmetric regime, where the nodes in each community reach an opposite average opinion. The transition from the asymmetric regime to the symmetric regime is shown to be discontinuous.

Keywords: random graphs, networks, critical phenomena of socio-economic systems, socio-economic networks

ArXiv ePrint: physics/0703266

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1742-5468/07/P08026+15$30.00

Contents

1. Introduction

In the last few years, the study of networks has received an enormous amount of attention from the scientific community [1, 2], in disciplines as diverse as computer and information sciences (the Internet and the World Wide Web), sociology and epidemiology (networks of personal or social contacts), biology (metabolic and protein interaction), etc. This outburst of interest has been driven mainly by the possibility to use networks in order to represent many complex systems and by the availability of communication networks and computers that allow us to gather and analyse data on a scale far larger than previously possible. The resulting change of scale (from a few dozen of nodes in earlier works to several thousands of nodes today) has not only led to the definition of new *statistical* quantities in order to describe *large* networks, e.g. degree distribution or clustering coefficient, but it has also addressed problems pertaining to statistical physics, for instance by looking at the interplay between the *microscopic* interactions of neighbouring nodes and the behaviour of the system at a *macroscopic* level. Such a problem takes place in social networks, i.e. nodes represent individuals and links between nodes represent their relations (e.g. friendship, co-authorship), when one tries to find macroscopic equations for the evolution of 'society'. Indeed, many studies have revealed non-trivial structures in social networks, such as fat-tailed degree distributions [3], a high clustering coefficient [4] and the presence of communities [5]. A primordial problem is therefore to understand how this underlying topology influences the way the interactions between individuals (physical contact, discussions) may (or may not) lead to collective phenomena. Typical examples would be the propagation of a virus [6], information [7, 8] or opinion [9, 10] in a social network, that may lead to the outbreak of an epidemic or of a new trend/fashion.

It is now well known that degree heterogeneity [6, 11, 12] is an important factor that may radically alter the macroscopic behaviour of a network but, surprisingly, the role

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played by its modular structure is still poorly known [13, 14]. It has been observed, though, that many social networks exhibit modular structures [5, 15, 16], i.e. they are composed of highly connected communities, while nodes in different communities are sparsely connected. This lack of interaction between communities certainly has consequences on the way information diffuses through the network, for instance, but it also suggests that nodes belonging to different communities may behave in a radically different way.

In this paper, we address such a problem by focusing on a simple model for networks with two communities, the coupled random networks (CRN). To do so, one consider a set of N nodes that one divides into two classes and one randomly assigns links between the nodes. Moreover, one assumes that the probability for two nodes to be linked is larger when they belong to the same class. Let us stress that CRN has been first introduced in [5] and that it is particularly suitable in order to highlight the role of the network modularity while preserving its randomness. The *microscopic* dynamics that we apply on CRN is the well-known majority rule (MR) [17]. MR is a very general model for opinion formation, i.e. nodes copy the behaviour of their neighbour, thereby suggesting that the results derived in this paper should also apply to other models of the same family. The effect of the inter-connectivity ν and of the frequency of random *flips*, measured by the parameter q ($\sim$*temperature*) on the phase diagram, is studied analytically. It is shown that three regimes may take place, depending on the parameters and on the initial conditions: a disordered regime, where no collective phenomena takes place; a symmetric regime, where the nodes in both communities reach the same average opinion; and an asymmetric regime, where the nodes in each community reach an opposite average opinion. The transition from the asymmetric regime to the symmetric regime is shown to be discontinuous. It is remarkable to note that a similar discontinuous transition also takes place when one applies MR to another network with communities, namely the coupled fully connected networks introduced in [13]. The main advantage of CRN is that its simpler structure allows us to perform all the calculations exactly and to consider the case of a non-vanishing q in detail.

2. Majority rule

The network is composed of N nodes, each of them endowed with an opinion that can be either α or β. At each time step, one of the nodes is randomly selected and two processes may take place. With probability q, the selected node randomly picks an opinion α or β, whatever its previous opinion or the opinion of its neighbours. With probability $1-q$, two neighbouring nodes of the selected node are also selected and the three agents in this *majority triplet* all adopt the state of the local majority (see figure 1). The parameter q therefore measures the competition between individual choices that have a tendency to randomize the opinions in the system, and neighbouring interactions that tend to homogenize the opinion of agents. In the case $q = 0$, it is well known that the system asymptotically reaches global consensus where all nodes share the same opinion [17]. In the other limiting case $q = 1$, the system is purely random and the average (over the realizations of the random process) number of nodes with opinion α at time t, denoted by A_t, goes to $N/2$.

Let us first focus on a network of individuals that are highly connected (in order to justify the use of mean-field methods) and where all the nodes are equivalent. That case

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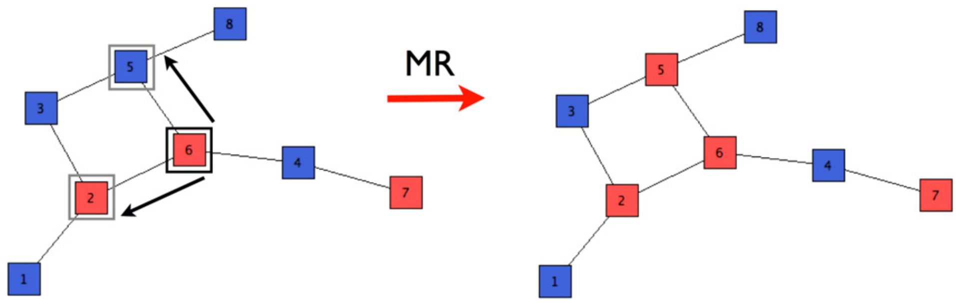

Figure 1. Sketch of a time step, where one node (surrounded in black) and two of its neighbours (surrounded in grey) are selected. The majority rule implies that the blue node becomes red.

has been studied in detail elsewhere [12] and is repeated here for the sake of clarity and introducing notation. It is straightforward to show that the mean-field rate equation for A_t is

$$A_{t+1} = A_t + q(\tfrac{1}{2} - a_t) - 3(1-q)a_t(1 - 3a_t + 2a_t^2), \tag{1}$$

where $a_t = A_t/N$ is the average proportion of nodes with opinion α. The term proportional to q, that accounts for the individual random flips, comes from the fact that the probability for the selected node to have an opinion β and to change its opinion is $(1-a_t)/2$ while the probability for a node α to change its opinion is $a_t/2$. Consequently, the contribution to the evolution of A_t is

$$\frac{1-a_t}{2} - \frac{a_t}{2} = \frac{1}{2} - a_t. \tag{2}$$

The second term, accounting for majority processes, is evaluated by calculating the probability that the majority triplet is composed of two nodes α and one node β, $3a_t^2(1-a_t)$, or of two nodes β and one node α, $3a_t(1-a_t)^2$. Let us also stress that configurations where the three nodes have the same opinion (α or β) do not contribute to the evolution of A_t, as nodes do not change their opinion in that case. The total contribution to the evolution of A_t is therefore

$$3\left(a_t^2(1-a_t) - a_t(1-a_t)^2\right) = -3a_t(1 - 3a_t + 2a_t^2). \tag{3}$$

It is easy to show that $a = 1/2$ is always a stationary solution of equation (1), as expected from symmetry reasons. $a = 1/2$ corresponds to a disordered state where no collective opinion has emerged in the system. It is useful to rewrite the evolution equation for the quantities $\Delta_t = A_t - N/2$ and $\delta_t = \Delta_t/N = a - 1/2$

$$\Delta_{t+1} = \Delta_t + \frac{\delta_t}{2}\left(3 - 5q - 12(1-q)\delta_t^2\right), \tag{4}$$

from which one finds that the disordered solution $a = 1/2$ ceases to be stable when $q < 3/5$. In that case, the system reaches one of the following ordered solutions:

$$\begin{aligned} a_- &= \frac{1}{2} - \sqrt{\frac{3-5q}{12(1-q)}} \\ a_+ &= \frac{1}{2} + \sqrt{\frac{3-5q}{12(1-q)}}. \end{aligned} \tag{5}$$

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The system therefore undergoes an order–disorder transition at $q = 3/5$. Under this value, a collective opinion has emerged due to the *imitation* between neighbouring nodes. In the limit case $q \to 0$, one finds $a_- = 0$ and $a_+ = 1$ in agreement with the results of [17].

Before going further, one should stress that the majority rule can be generalized by picking $G - 1$ (with G an odd number) neighbours of the selected node instead of only two [17], so that the set of interacting nodes is a quintet, septet, etc., instead of a triplet. In that case, it is straightforward to generalize the above calculations and to show that

$$A_{t+1} = A_t + q\left(\frac{1}{2} - a_t\right) + (1-q)\sum_{k=1}^{(G-1)/2} k\frac{G!}{k!(G-k)!}(a_t^{G-k}(1-a_t)^k - a_t^k(1-a_t)^{G-k}), \tag{6}$$

where $G!/k!(G-k)!$ is the number of ways to select k nodes among G ones and where the factor k is due to the fact that the k nodes belonging to the minority change their opinion. One observes that this system exhibits an order–disorder transition for any odd value of G and that the location of the transition $q_c(G)$ increases when G is increased. For instance, when $G = 5$, the ordered solution is

$$a_- = \frac{1}{2} - \sqrt{\frac{5}{12} - \frac{\sqrt{5 - 4q - q^2}}{6\sqrt{5}(1-q)}}, \tag{7}$$

with $a_+ = 1 - a_-$ and the transition now takes place at $q_c(5) \approx 0.814$. This shift of q_c may be understood by noting that the order–disorder transition comes from a competition between the frequency of random flips $\sim q$ and the frequency of interaction $\sim(1-q)$. When the number of interacting nodes is increased, the frequency of interaction for each node is also increased (while the frequency of random flips remains the same), so that q_c should increase. From now on, we restrict the scope to the case $G = 3$ for the sake of simplicity.

3. Role of communities

3.1. Coupled random networks

Distinct communities within networks are defined as subsets of nodes which are more densely linked when compared to the rest of the network. For the sake of simplicity, we restrict the scope to networks composed of only two communities, denoted by 1 and 2. Our goal is to build an uncorrelated random network where nodes in 1 are more likely to be connected with nodes in 1 than with nodes in 2, and vice versa. To do so, we consider a set of N nodes that we divide nodes into two classes, 1 and 2. We evaluate each pair of nodes in the system and draw a link between these nodes with probability p_{ij}, where i and $j \in \{1, 2\}$ are the class to which the two nodes belong. In the following, we consider a case where the number of nodes in 1 and 2, respectively denoted by N_1 and N_2, are equal: $N_1 = N_2 = N/2$. Moreover, we restrict the scope to the following probabilities: $p_{12} = p_{21} = p_{\text{cross}}$ and $p_{11} = p_{22} = p_{\text{in}}$. By construction, nodes in 1 are therefore connected on average to $k_{\text{in}} = p_{\text{in}}(N-1)/2 \approx p_{\text{in}}N/2$ nodes in 1 and to $k_{\text{cross}} = p_{\text{cross}}N/2$ nodes in 2, while nodes in 2 are connected to k_{cross} nodes in 1 and k_{in} nodes in 2. Let us stress that a similar model has already been used in order to test methods for community detection in [5].

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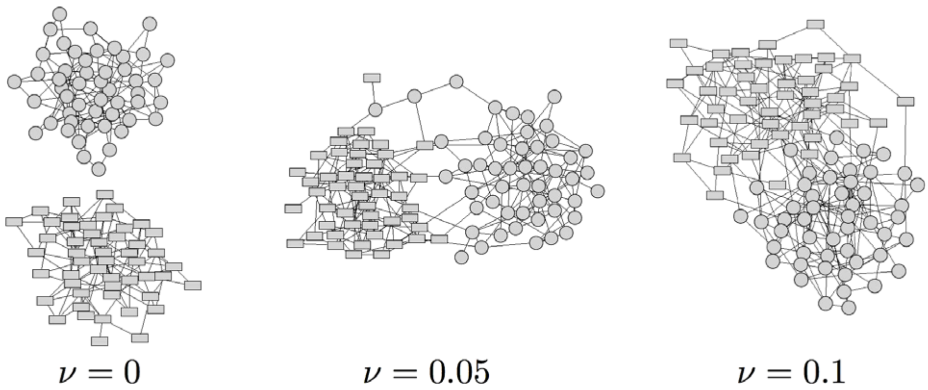

Figure 2. Typical realizations of coupled random networks for small values of ν. The network is composed of $N = 100$ nodes and $p_{\text{in}} = 0.1$. The system obviously consists of two communities that are less and less discernible for larger values of ν. We have assigned different shapes (circles and rectangles) to nodes belonging to different communities. The graphs were plotted thanks to the *visone* graphical tools [18].

This binary mixture (see figure 2), which we call a *coupled random network* (CRN), is particularly suitable in order to reveal the role of network modularity [5]. Indeed, the interconnectivity between the communities is tunable through the parameter $\nu = p_{\text{cross}}/p_{\text{in}}$. In the following, we focus on the interval $\nu \in [0, 1]$ for which inter-community links are less frequent than intra-community links. When $\nu \to 1$, one recovers an homogeneous network where all nodes are *a priori* equivalent while, when $\nu \ll 1$, the communities are sparsely connected with each other. Before going further, one should also point to an alternative model of modular networks, the coupled fully connected network introduced in [13]. MR has been applied on this topology with $q = 0$ and it has been shown that a discontinuous transition from an asymmetric to a symmetric state takes place. In the following, we will not only show that a similar discontinuous transition takes place on CRN, but we will also study analytically the behaviour of the system for $q \neq 0$.

3.2. Equation of evolution

Let us denote by A_1 and A_2 the average number of nodes with opinion α among the two types of nodes. Let us first focus on the contributions when majority triplets are selected. At each time step, the probability that the selected node belongs to the first community is $1/2$. In that case, the probability that a randomly chosen link around the selected node goes to a node in 1 is $k_{\text{in}}/(k_{\text{in}} + k_{\text{cross}}) = 1/(1 + \nu)$. The probability that this randomly chosen link goes to a node in 2 is $k_{\text{cross}}/(k_{\text{in}} + k_{\text{cross}}) = \nu/(1 + \nu)$. Consequently, the probability that the selected node belongs to 1 and that both of its selected neighbours belong to 2 is

$$\frac{1}{2}\frac{\nu^2}{(1+\nu)^2}. \tag{8}$$

Similarly, the probability that the selected node belongs to 1, that one of its neighbours belongs to 1 and that the other neighbour belongs to 2 is

$$\frac{1}{2}\frac{2\nu}{(1+\nu)^2}, \tag{9}$$

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while the probability that all three nodes belong to 1 is

$$\frac{1}{2}\frac{1}{(1+\nu)^2}. \tag{10}$$

The probabilities of events when the selected node belongs to 2 are found in the same way. Putting all contributions together, one finds the probabilities $P_{(x,y)}$ that x nodes 1 and y nodes 2 belong to the majority triplet

$$\begin{aligned}
P_{(3,0)} &= \frac{1}{2}\frac{1}{(1+\nu)^2}\\
P_{(2,1)} &= \frac{1}{2}\frac{2\nu}{(1+\nu)^2} + \frac{1}{2}\frac{\nu^2}{(1+\nu)^2} = \frac{1}{2}\frac{\nu^2+2\nu}{(1+\nu)^2}\\
P_{(1,2)} &= \frac{1}{2}\frac{\nu^2+2\nu}{(1+\nu)^2}\\
P_{(0,3)} &= \frac{1}{2}\frac{1}{(1+\nu)^2}
\end{aligned} \tag{11}$$

where the normalization $\sum_{xy} P_{(x,y)} = 1$ is verified. In order to derive coupled equations for $A_{1;t}$ and $A_{2;t}$ that would generalize equation (1), one needs to evaluate the evolution of these quantities when a triplet (x, y) is selected. To do so, one follows the steps described in [13] and, when $q = 0$, one obtains the equation of evolution

$$\begin{aligned}
A_{1;t+1} - A_{1;t} &= \frac{3}{2}\frac{1}{(1+\nu)^2}(a_1^2 b_1 - a_1 b_1^2) + \frac{\nu^2+2\nu}{(1+\nu)^2}(a_2 a_1 b_1 - a_1 b_2 b_1)\\
&\quad + \frac{1}{2}\frac{\nu^2+2\nu}{(1+\nu)^2}(a_2^2 b_1 - a_1 b_2^2)\\
A_{2;t+1} - A_{2;t} &= \frac{3}{2}\frac{1}{(1+\nu)^2}(a_2^2 b_2 - a_2 b_2^2) + \frac{\nu^2+2\nu}{(1+\nu)^2}(a_1 a_2 b_2 - a_2 b_1 b_2)\\
&\quad + \frac{1}{2}\frac{\nu^2+2\nu}{(1+\nu)^2}(a_1^2 b_2 - a_2 b_1^2),
\end{aligned} \tag{12}$$

where a_i and b_i are, respectively, the proportion of nodes with opinion α and β in the community i. After incorporating the term due to random flips, proportional to q, one obtains the set of nonlinear equations

$$\begin{aligned}
A_{1;t+1} - A_{1;t} &= \frac{q}{4} - \frac{q a_1}{2} + (1-q)\left[\frac{3}{2}\frac{1}{(1+\nu)^2}(a_1^2 b_1 - a_1 b_1^2)\right.\\
&\quad \left. + \frac{\nu^2+2\nu}{(1+\nu)^2}(a_2 a_1 b_1 - a_1 b_2 b_1) + \frac{1}{2}\frac{\nu^2+2\nu}{(1+\nu)^2}(a_2^2 b_1 - a_1 b_2^2)\right]\\
A_{2;t+1} - A_{2;t} &= \frac{q}{4} - \frac{q a_2}{2} + (1-q)\left[\frac{3}{2}\frac{1}{(1+\nu)^2}(a_2^2 b_2 - a_2 b_2^2)\right.\\
&\quad \left. + \frac{\nu^2+2\nu}{(1+\nu)^2}(a_1 a_2 b_2 - a_2 b_1 b_2) + \frac{1}{2}\frac{\nu^2+2\nu}{(1+\nu)^2}(a_1^2 b_2 - a_2 b_1^2)\right].
\end{aligned} \tag{13}$$

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Direct calculations show that equation (13) reduces to equation (1) in the limit $\nu = 1$, as expected due to the indistinguishability of the nodes in that case.

3.3. Stability of the disordered solution

It is straightforward to show that $a_1 = 1/2$, $a_2 = 1/2$ is always a stationary solution of equation (13), whatever the values of ν and q. This solution consists of a disordered state where both communities behave similarly and where no favorite opinion has emerged due to MR. We study the stability of this disordered state by looking at small deviations $\epsilon_1 = a_1 - 1/2$ and $\epsilon_2 = a_2 - 1/2$ and keeping only linear corrections. In the continuous time limit and after rescaling the time units, the evolution equations for these deviations are

$$\begin{aligned}\partial_t \epsilon_1 &= -\frac{-3 + 5q + 2\nu(1+q) + \nu^2(1+q)}{4(1+\nu)^2}\epsilon_1 + \frac{\nu(2+\nu)(1-q)}{(1+\nu)^2}\epsilon_2 \\ \partial_t \epsilon_2 &= \frac{\nu(2+\nu)(1-q)}{(1+\nu)^2}\epsilon_1 - \frac{-3 + 5q + 2\nu(1+q) + \nu^2(1+q)}{4(1+\nu)^2}\epsilon_2.\end{aligned} \tag{14}$$

The eigenvalues of this linearized matrix of evolution are

$$\begin{aligned}\lambda_1 &= \frac{(3 - 10\nu - 5\nu^2 - 5q + 6\nu q + 3\nu^2 q)}{4(1+\nu)^2} \\ \lambda_2 &= \tfrac{1}{4}(3 - 5q).\end{aligned} \tag{15}$$

By definition, the disordered solution is stable only when both eigenvalues are negative [19], thereby ensuring that a small perturbation asymptotically vanishes. It is easy to show that only the values of q in the interval $]3/5, 1]$ respect this condition, whatever the value of ν. This implies that the location of the order–disorder transition is not affected by the modularity of the network. Let us also stress that λ_1 goes to λ_2 when $\nu = 0$. This is expected, as the system is composed of two independent networks in that case, so that the equations of evolution for A_1 and A_2 are decoupled.

3.4. Symmetric solution

Our knowledge of the case $\nu = 1$ (section 2) and the natural symmetry of CRN suggests to look at solutions of the form $a_1 = 1/2 + \delta_S$, $a_2 = 1/2 + \delta_S$ (S for symmetric). By inserting this ansatz into equation (12)

$$\begin{aligned}\frac{q}{4} - \frac{q(1/2+\delta_S)}{2} + (1-q)\Bigg[\frac{3}{2}\frac{(1/2+\delta_S)^2(1/2-\delta_S) - (1/2-\delta_S)^2(1/2+\delta_S)}{(1+\nu)^2} \\ + \frac{\nu^2+2\nu}{(1+\nu)^2}\left(\left(\frac{1}{2}+\delta_S\right)^2\left(\frac{1}{2}-\delta_S\right) - \left(\frac{1}{2}-\delta_S\right)^2\left(\frac{1}{2}+\delta_S\right)\right) \\ + \frac{1}{2}\frac{\nu^2+2\nu}{(1+\nu)^2}\left(\left(\frac{1}{2}+\delta_S\right)^2\left(\frac{1}{2}-\delta_S\right) - \left(\frac{1}{2}-\delta_S\right)^2\left(\frac{1}{2}+\delta_S\right)\right)\Bigg] = 0,\end{aligned} \tag{16}$$

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direct calculations lead to the relation

$$-\frac{q\delta_{\rm S}}{2}+\left(\frac{\delta_{\rm S}}{2}-2\delta_{\rm S}^3\right)\frac{(1-q)}{(1+\nu)^2}\left[\frac{3}{2}+\frac{3}{2}\nu^2+3\nu\right]=0. \tag{17}$$

Let us insist on the fact that equation (17) is exact and not an expansion for small $\delta_{\rm S}$. It is a direct consequence of the above *symmetry* ansatz. The disordered solution $\delta_{\rm S}=0$ obviously satisfies equation (17), but symmetric solutions are also possible if they satisfy

$$-\frac{q(1+\nu)^2}{2(1-q)}+\left(\frac{1}{2}-2\delta_{\rm S}^2\right)\left[\frac{3}{2}+\frac{3}{2}\nu^2+3\nu\right]=0, \tag{18}$$

so that symmetric solutions have the form

$$\delta_{\rm S}^2=\frac{3-5q}{12(1-q)}. \tag{19}$$

It is remarkable to note that the symmetric solution does not depend on the inter-connectivity ν. This is checked by comparing (19) with the solution (5) obtained when the system is composed of only one community. It is also straightforward to show that (19) is stable when $q<3/5$, as expected, from which one concludes that none of the characteristics of the order–disorder transition have been altered by the modularity of the network.

3.5. Asymmetric solution

Let us first focus on the case $q=0$ where the dynamics is only driven by MR. We have shown above that the system may reach a symmetric state $a_1=1$, $a_2=1$ or $a_1=0$, $a_2=0$ in that case (see equation (19)). However, computer simulations (figures 3(a) and (b)) show that an asymmetric stationary state may prevail for small enough values of ν. Computer simulations also show that the asymmetric state is characterized by averages of the form $a_1=1/2+\delta_{\rm A}$ and $a_2=1/2-\delta_{\rm A}$ (A for asymmetric). Such solutions are distinguished from the symmetric solutions by focusing on the order parameter $|a_1-a_2|$ that is equal to $|a_1-a_2|=2\delta_{\rm A}\neq 0$ in the asymmetric state, while $|a_1-a_2|$ vanishes when both communities reach the same opinion. Based on these numerical results, we look for stationary solutions of equation (12) having this form. It is straightforward to show that the equations for A_1 and A_2 lead to the following condition:

$$(3\delta_{\rm A}-12\delta_{\rm A}^3)+(\nu^2+2\nu)(-2\delta_{\rm A}+8\delta_{\rm A}^3)-(\nu^2+2\nu)(3\delta_{\rm A}+4\delta_{\rm A}^3)=0, \tag{20}$$

whose solutions are either $\delta_{\rm A}=0$ (disordered state) or

$$\delta_{\rm A}^2=\frac{3-5(\nu^2+2\nu)}{12-4(\nu^2+2\nu)}. \tag{21}$$

The *disordered* solution $\delta_{\rm A}=0$ is unstable when $q=0$, as shown above, and it is thus discarded. One should also stress that the *symmetric* solution (19) is frozen when $q=0$, i.e. all the nodes are either α or β so that no dynamics is possible when a majority triplet is selected, while fluctuations continue to take place in the *asymmetric* solution (21).

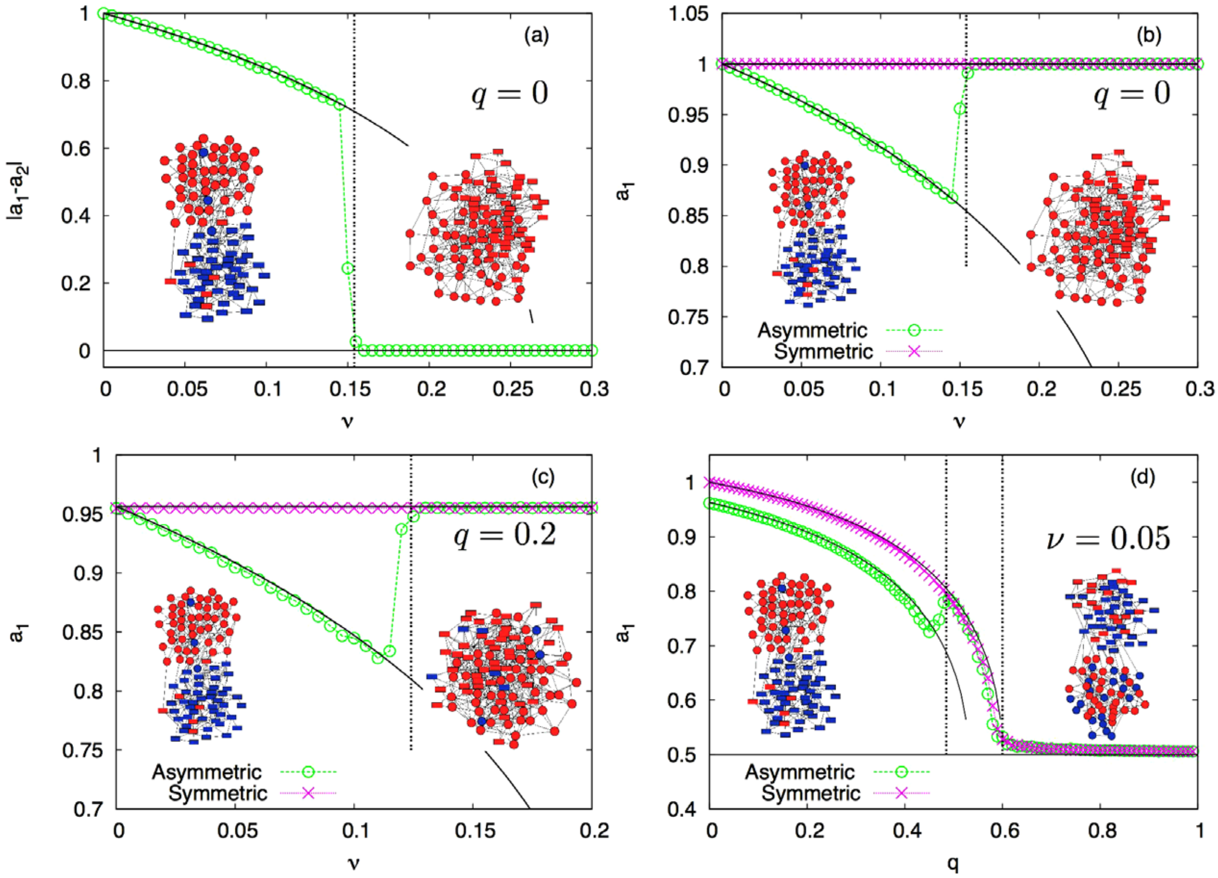

Figure 3. Computer simulations of MR on coupled random networks with $N = 10^4$ nodes and $p_{\rm in} = 0.01$. The simulations are stopped after 10^3 steps/node and the results averaged over 100 realizations of the random process. The vertical dashed lines point to the theoretical transition value obtained from equation (28) and to the critical value $q = 3/5$ (d). The solid lines correspond to the theoretical predictions (19) and (21). The simulations are either started with a symmetric initial condition $a_1 = 1$, $a_2 = 1$ or with an asymmetric initial condition $a_1 = 1$, $a_2 = 0$. (a) Bifurcation diagram of $|a_1 - a_2|$ as a function of ν, for simulations starting from asymmetric initial conditions. The system ceases to be asymmetric above $\nu_{\rm c} \approx 0.15$. (b) and (c) Bifurcation diagram of a_1 as a function of ν, starting the simulations from asymmetric or symmetric initial conditions for $q = 0$ (b) and $q = 0.2$ (c). The systems behave qualitatively in the same way when $q = 0$ and $q = 0.2$, except that the symmetric state is frozen when $q = 0$ (all the nodes have and keep the same opinion, $a_1 = 1$) while a_1 fluctuates around its asymptotic value when $q \neq 0$. (d) Bifurcation diagram of a_1 as a function of q, starting the simulations from asymmetric or symmetric initial conditions for $\nu = 0.05$. In that case, the system may undergo two transitions: one from the asymmetric to the symmetric state at $q \approx 0.485$ and one from the symmetric to the disordered state at $q = 3/5$.

By construction, the asymmetric solution exists only when $\delta_{\rm A}^2 \geq 0$, namely when $\nu \in [0, (-5 + \sqrt{40})/5 \approx 0.26]$. In order to check the stability of (21) in this interval, we focus on the small deviations $\epsilon_1 = a_1 - (1/2 + \delta_{\rm A})$ and $\epsilon_2 = a_2 - (1/2 - \delta_{\rm A})$. After inserting

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these expressions into equation (12) and keeping only linear terms, lengthy calculations lead to the eigenvalues

$$\begin{aligned}\lambda_1 &= -\frac{3-10\nu-5\nu^2}{2(1+\nu)^2}\\ \lambda_2 &= -\frac{3}{2}\frac{1-6\nu-3\nu^2}{(1+\nu)^2},\end{aligned} \tag{22}$$

from which one shows that the asymmetric solution loses its stability at a critical value

$$\nu_c = (\sqrt{48}-6)/6 \approx 0.15, \tag{23}$$

The transition is easily shown to be discontinuous at ν_c, because the asymmetric solution is $1/2+\delta_A(\nu_c) \approx 0.85$ at that point while the symmetric solution is always equal to 1 when $q=0$ (see equation (19)). When $\nu < \nu_c$, the system may reach either the symmetric or the asymmetric solution depending on the initial conditions (and on the fluctuations). When $\nu > \nu_c$, in contrast, only the symmetric solution is attained in the long time limit. As stressed before, MR also undergoes a discontinuous transition from an asymmetric state to a symmetric state when it is applied to coupled fully connected networks [13]. This similarity therefore suggests that such a transition is a generic feature of networks with modular structure. Let us also note that the above theoretical results have been verified by performing computer simulations of MR. The asymmetric solution (21) and the location of the transition (23) are in perfect agreement with the simulations (figures 3(a) and (b)).

It is straightforward to generalize these results when $q > 0$. To do so, one inserts $a_1 = 1/2+\delta_A$, $a_2 = 1/2-\delta_A$ into equation (13) from which one obtains the relation

$$\begin{aligned}-2q\delta_A + \frac{(1-q)}{(1+\nu)^2}&[(3\delta_A - 12\delta_A^3) + (\nu^2+2\nu)(-2\delta_A + 8\delta_A^3)\\ &- (\nu^2+2\nu)(3\delta_A + 4\delta_A^3)] = 0.\end{aligned} \tag{24}$$

The stationary solutions of (24) are either $\delta_A = 0$ or

$$\delta_A^2 = \frac{3-2(q(1+\nu)^2/(1-q)) - 5(\nu^2+2\nu)}{12-4(\nu^2+2\nu)}, \tag{25}$$

and the eigenvalues of the linearized equation of evolution around the stationary solution (25) are

$$\begin{aligned}\lambda_1 &= -\frac{3-10\nu-5\nu^2-5q+6\nu q+3\nu^2 q}{2(1+\nu)^2}\\ \lambda_2 &= -\frac{3-18\nu-9\nu^2-5q+14\nu q+7\nu^2 q}{2(1+\nu)^2}.\end{aligned} \tag{26}$$

The asymmetric solution is shown to lose its stability when

$$3-18\nu-9\nu^2-5q+14\nu q+7\nu^2 q = 0 \tag{27}$$

that one simplifies into

$$\nu_c(q) = -1 + 2\sqrt{\frac{3q-3}{7q-9}}. \tag{28}$$

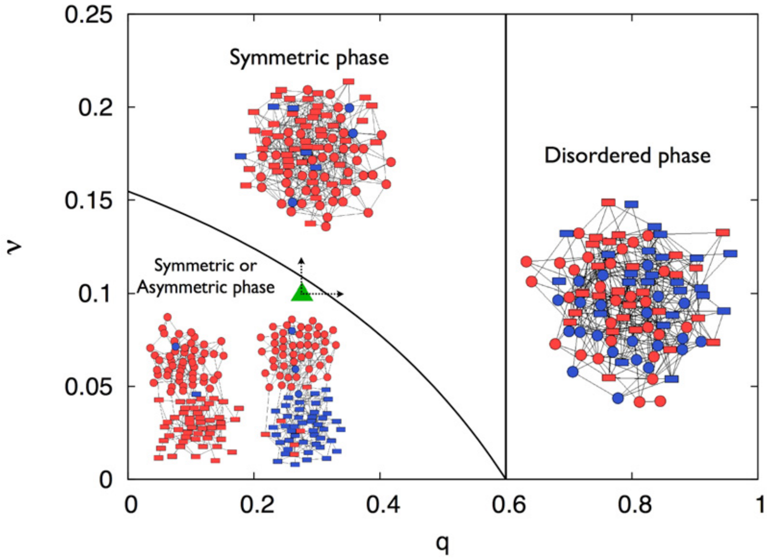

Figure 4. Phase diagram of MR on CRN. Three phases may take place: (i) a disordered phase when $q > 3/5$; (ii) a symmetric phase when $q < 3/5$; (iii) an asymmetric phase when $q < 3/5$ and when $\nu < -1 + 2\sqrt{(3q-3)/(7q-9)}$. A system in the asymmetric state, but close to the transition line, e.g. the green triangle, may lose its stability due to an increase of the number of inter-community links (along ν) or to an increase of the internal fluctuations (along q). The asymmetric state differs from the symmetric and from the disordered states by the fact that it exhibits correlations between the shape of the nodes (i.e. the community to which they belong) and their colour (i.e. their opinion).

This relation therefore determines the critical line above which only symmetric solutions prevail (see figure 4). One can shown that (28) decreases with q and that it goes to zero at the transition point $q = 3/5$. It is also easy to show that the transition from the asymmetric to symmetric state is discontinuous for any values of $q < 3/5$. Equations (25) and (28) have been successfully checked by computer simulations, as shown in figure 3(c) (figure 3(d)), where one plots the asymptotic value of a_1 as a function of ν (of q) at a fixed value of $q = 0.2$ (of $\nu = 0.05$).

Let us now emphasize some points that deserve attention. First, one should note that the existence of asymmetric solutions is expected in the limit $\nu \to 0$, where the two sub-clusters are completely disconnected, $k_{\mathrm{cross}} = 0$, and evolve independently from each other. In that case, both sub-networks may reach internal consensus if $q < 3/5$ and there is a probability 1/2 that the opinion in the first community is the same as in the second community, while these opinions differ otherwise. This interpretation is confirmed by noting that equation (25) is identical to (5) when $\nu = 0$. We have shown in this section that asymmetric solutions may prevail for values of ν and q such that $\nu < \nu_{\mathrm{c}}(q)$, and not only when $\nu = 0$, $q < 3/5$. It is also interesting to stress that a system in the asymmetric state, but close to the transition line, e.g. the green triangle in figure 4, may lose its stability due to an increase of the number of inter-community links (along ν) or to an increase of the internal fluctuations (along q). Moreover, once the system has reached the symmetric phase, it will remain in it forever, even if the

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system parameters return below the transition line ($\nu < \nu_c(q)$). This is due to the fact that a symmetric state is reached for such parameters when the initial condition is itself symmetric. Consequently, diversity may irreversibly disappear from a network that is too close from the transition line. Finally, let us stress that a system whose parameters verify $\nu < \nu_c(q)$ may undergo two transitions when q is increased (at fixed values of ν), a first transition from the asymmetric to the symmetric state when $\nu = \nu_c(q)$ and another transition from the symmetric state to the disordered state at $q = 3/5$ (see figure 3(d)).

3.6. Community detection

In this section, we would like to discuss a possible application of this work in the context of community detection. As emphasized in the introduction, many networks exhibit modular structures. This is true for social networks, where groups of nodes correspond to social communities or cliques, but also for the World Wide Web, where groups of highly connected sites may be related by a common topic [20, 21], biological networks, where clusters of nodes may be associated to functional modules [22]–[24] and even self-citation networks, where clusters may reveal the field mobility of a scientist [25]. It is therefore of significant practical importance to find efficient methods in order to identify modular structures in large graphs [26]–[33]. Amongst others investigations, it has been shown that the use of Ising-like models may be helpful in order to unravel communities in complex networks [34, 35]. This approach consists in applying a dynamics such as the majority rule on a pre-given network and to identify a community as a set of nodes having the same opinion/spin. In the language of this paper, the identification of the communities is possible only if the system reaches an asymmetric regime, i.e. if both communities reach different states. Our work could therefore provide a theoretical background for the use of such identification techniques. For instance, our results show that MR has a 'minimum resolution', i.e. MR does not discriminate communities when $\nu > \nu_c(q)$ (with $\nu_c(q) < (\sqrt{48}-6)/6$) because the asymmetric state is not stable in that case. One expects that such a 'minimum resolution' should also take place when other models of opinion formation are applied to a modular network.

4. Conclusion

In this paper, we have focused on coupled random networks (CRN) in order to show how the presence of communities affects the emergence of collective phenomena on complex networks. To do so, we have applied an opinion-formation model, i.e. the majority rule (MR), on CRN and we have studied analytically the effect of the inter-connectivity ν between the communities on the phase diagram. It is shown that three kinds of asymptotic states may take place, depending on the parameters and on the initial conditions. The disordered state, where no collective phenomena emerge, and the symmetric state, where a global consensus is reached, are well-known solutions of MR. However, due to its modularity, the system may also exhibit an asymmetric state, where nodes belonging to different communities exhibit different behaviours. The transition from the disordered state to the symmetric state is shown to be continuous, while it is discontinuous from the symmetric to the asymmetric state. Let us emphasize that such asymmetric states have been observed in many situations. Amongst others, one may think

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of sub-communities in collaboration networks, that may correspond roughly to topics of research [5], the Political Blogosphere, where it was observed that bloggers having a different political opinion are segregated [36], the existence of niche markets [37], where small communities may use products that are different from those used by the majority, language dynamics [38, 39], where it is well known that natural frontiers may also coincide with a linguistic frontier [40], etc. The discontinuity of the transition from this asymmetric state to a symmetric 'globalized' state might thus have radical consequences for systems close to the transition line: the addition of a few links between the communities or a small increase of the fluctuations inside the system (see figure 4) may be sufficient in order to drive the system out of the asymmetric state. Such rapid shifts of behaviour of a whole sub-community should be searched in empirical data, in the dynamics of trends or fashion [41] for instance.

To conclude, we would like to point to a possible generalization that would certainly make the model more realistic. In this paper, we have focused on a model of opinion formation evolving on a static topology. The effect of the network structure on the formation of a collective opinion has therefore been highlighted, but the fact that the opinion of the nodes themselves might influence the topology around them, e.g. links between nodes with different opinions could disappear, has not been considered. In this direction, it is interesting to point to recent models from evolutionary game theory [42]–[45], where individuals may adjust their social links based on their self-interest [46]. This co-evolution of the agent's strategy and of their links has been shown to change radically the way the system self-organizes [46]. In the case of opinion formation, a model where the network inter-connectivity ν might co-evolve [47, 48] with the node spins/opinions would thus also be of high interest. One should also emphasize that the coupled random networks studied in this paper are composed of only two communities and that they do not exhibit well-known properties of social networks such as their degree heterogeneity (see [12] for a study of MR on networks with degree heterogeneity) or their non-trivial correlations, e.g. clustering coefficient [4] and assortativity [49]. A generalization of our analytical approach to more than two communities is in principle straightforward and one expects that similar behaviours should take place, e.g. emergence of asymmetric states. In contrast, the incorporation of realistic correlations between nodes would enormously complicate calculations and would probably make an analytical description impossible, so that numerical simulations would be needed in that case. Other interesting generalizations would incorporate agents with ageing [50]–[52] or memory [53], the presence of leaders [54], etc.

Acknowledgments

This work has been supported by European Commission Project CREEN FP6-2003-NEST-Path-012864. RL thanks J Hołyst, M Buchanan and A Scharnhorst for fruitful comments.

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